Normalized Exponential Tilting: Pricing and Measuring Multivariate Risks

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Abstract: This paper discusses methods of risk neutralizing statistical distributions by applying exponential tilting of the probability density of a risk *X*, with respect to a reference risk *Y*. It proposes a normalization procedure based on percentile matching to convert the reference risk *Y* to a standard normal variable Z. The resulting normalized exponential tilting extends classic theories of pricing risks, including CAPM and the Black-Merton-Scholes. The paper then extends normalized exponential tilting to multivariate risks, establishing a link to multivariate probability distortions. The paper provides efficient routines for computing risk-neutralized multivariate probability distributions and illustrative examples of pricing contingent claims on multiple risks.

Key words: Risk neutralization; Pricing in Incomplete Markets; Risk Measurement; Exponential tilting; Wang transform, Esscher transform

1. Introduction

Change of probability measure is a common theme in pricing and valuation of risks and contingent claims. In no-arbitrage financial pricing theory, the price of a contingent claim is evaluated as the expected payoff under a risk neutral probability measure that is different from its statistical counterpart (Harrison and Kreps, 1979). In a complete market, the risk neutral probability measure can be readily inferred from market transaction data. In an incomplete market, however, we do not have sufficient market data to infer a risk-neutral distribution; instead, we have historical data which allows us to estimate statistical distribution of the potential outcomes and their respective likelihoods. The question then arises as to how to construct a risk neutral density from the estimated statistical density, as a basis for pricing contingent claims written on the underlying risks. Madan and Unal (2004) studied this problem and referred to this change of measure as "risk neutralizing" the statistical distribution.

Exponential tilting, as a general method for neutralizing the statistical distribution, has been discussed by many authors (see Buhlmann, 1980; Gerber and Shiu, 1996; Madan and Unal; 2004; among many others). As Madan and Unal (2004) put it, exponential tilting is broadly consistent with much of the current literature on on-arbitrage pricing of contingent claims (Duffie, 1992; Heston, 1993; Karatzas and Shreve, 1991; Gerber and Shiu, 1996), and is potentially widely applicable in pricing risks embedded in loan defaults, mortgage refinancing, electricity trading, weather derivatives, and catastrophic insurance.

This paper (in Sections 2-5) starts by defining *exponential tilting* of the probability density function of X with respect to a reference variable Y. What makes this paper unique from the previous ones is by introducing a normalization procedure on the reference Y via percentile mapping to a standard normal variable Z. It shows that *normalized exponential tilting* of the probability density function of X (with respect to Z) is equivalent to applying the Wang transform to the cumulative distribution of X, and is an extension of the Capital Asset Pricing Model to risks with general-shaped distributions.

The need for changing multivariate probability measures arises in pricing of contingent claims on multiple underlying assets or liabilities (and when allocating total company risk capital to various business units). The second part of this paper (sections 6-9) extends the normalized exponential tilting to multivariate cases. It gives efficient routines for computing the risk-neutralized multivariate probability distribution, and provides examples in pricing contingent claims on multiple risks.

2. Normalized Exponential Tilting

We shall consider *risks* that are random variables in some probability space (Ω, P). For any random variable *X*, we let F_X represent its cumulative distribution function (CDF). We let f_X represent the probability density function (p.d.f) of *X* (in the discrete case, the p.d.f. is also referred to as the probability function).

Consider two risks *X* and *Y*. Assume that X is absolutely continuously with respect to Y; that is, $f_Y(x) > 0$ for all points *x* with $f_X(x) > 0$.

Definition 2.1 For each scenario ω in the probability space (Ω, P), the exponential tilting of *X* with respect to *Y* is defined by the following risk-neutralization:

(2.1) $$f_X^*\left(x(\omega)\right) = c \cdot f_X\left(x(\omega)\right) \cdot \exp\left(\lambda \cdot y(\omega)\right),$$

where f_X and f_X^* represent the p.d.f. for *X*, before and after the exponential tilting, respectively, and *c* is a normalizing coefficient. The real-valued parameter λ in equation (2.1) controls the magnitude of risk-adjustment.

In terms of probability density function, the exponential tilting of *X* with respect to *Y* can be written as follows:

(2.2) $$f_X^*(x) = f_X(x) \cdot \frac{E\left[\exp(\lambda Y) \mid X = x\right]}{E\left[\exp(\lambda Y)\right]}.$$

The ratio

$$RN(x) = \frac{f_X^*(x)}{f_X(x)} = \frac{E\left[\exp(\lambda Y) \mid X = x\right]}{E\left[\exp(\lambda Y)\right]},$$

gives the Radon-Nikodym derivative of f_X^* w.r.t. f_X. This ratio is also called the measure-change density as in Madan and Unal (2004).

In the context of an economic model for optimal risk exchange, Buhlmann (1980) derived that the Pareto-optimal equilibrium price for a risk *X* can be represented as the expected value of the risk-neutralized distribution as in equation (2.1), whereas the reference *Y* represents portfolio aggregate risk.

In the special case of *Y*=*X*, equation (2.1) defines an exponential tilting of *X* w.r.t. itself:

$$f_X^*(x) = f_X(x) \cdot \frac{\exp(\lambda x)}{E[\exp(\lambda X)]}.$$

The relation is also widely known as the Esscher transform. Gerber and Shiu (1996) applied the Esscher transform in pricing options.

With the exponential tilting in equation (2.1), we do not have a consistent interpretation of the λ parameter, except for in the special case when *Y* is a normal (Gaussian) variable. Indeed, if we keep the value of λ fixed, the scale and shape of the reference variable *Y* can significantly impact the result of exponential tilting.

In order to get a consistent interpretation of λ, we propose a normalization procedure of the reference variable *Y* through percentile-matching to a standard normal variable *Z*. In other words, $Y = F_Y^{-1}(\Phi(Z))$, with Φ being the CDF of Z, and $F_Y^{-1}(p) = \inf\{y | F_Y(y) \geq p\}$. We shall call Z as a normalized variable of *Y*. Next we use *Z* to replace the reference variable *Y* in the exponential tilting.

Definition 2.2 Let *Z* be a normalized variable of the reference *Y*. We define a **normalized exponential tilting** of *X* with respect to reference *Y* as follows:

(2.3) $$f_X^*(x) = f_X(x) \cdot \frac{E[\exp(\lambda Z) \mid X = x]}{E[\exp(\lambda Z)]}.$$

To keep the notations straight, we summarize the above as follows:

- Before: Exponential tilting of *X* w.r.t. *Y*
- Normalization: Percentile mapping *Y* to a standard normal variable *Z*
- After: Normalized exponential tilting of *X* w.r.t. *Z*

Note that here Z essentially replaces Y as a *normalized* reference variable.

The rationales and benefits of introducing the above normalization procedure will be given in Section 4.

3. Probability Distortions

Now let us introduce probability distortions, as a different method of changing probability measures, and as a general class of coherent measures of risk (including the conditional tail expectation measure as advocated by Artzner et al, 1999).

Definition 3.1. Let g:[0, 1]→[0, 1] be a differentiable function with g(0)=0 and g(1)=1. Given the CDF $F(x)$ for a random variable X, the transformed CDF

(3.1) $$F_X^*(x) = g\big(F(x)\big),$$

defines a risk-neutralized probability measure. The probability transform in equation (3.1) is called a *probability distortion*.

The probability distortion in equation (3.1) yields the following Radon-Nikodym derivative:

- In the discrete case where X takes on values $\{x_1, \cdots, \quad x_{i-1}, \quad x_i, \quad \cdots\}$:

$$RN_g(x_i) = \frac{f_X^*(x_i)}{f_X(x_i)} = \frac{g\big(F_X(x_i)\big) - g\big(F_X(x_{i-1})\big)}{F_X(x_i) - F_X(x_{i-1})}$$

- In the continuous case where X has a positive probability density at x:

$$RN_g(x) = g'\big(F_X(x)\big).$$

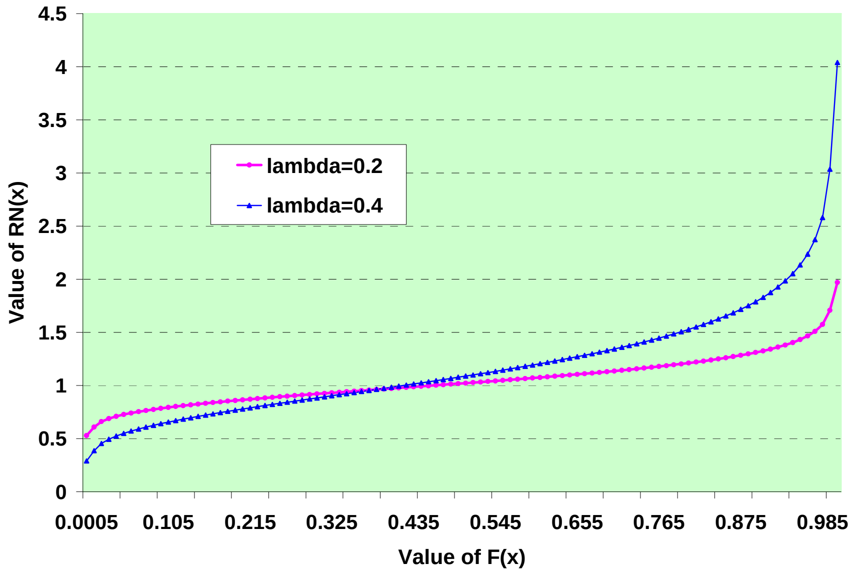
The following family of distortion corresponds to the Wang transform (see Wang, 2000; Dowd, 2005):

(3.2) $$F_X^*(x) = g(F_X(x)) = \Phi\left[\Phi^{-1}(F_X(x)) - \lambda\right].$$

In the case that X is a continuous variable, the Wang transform corresponds to the following Radon-Nikodym derivative (also see Figure 3.1):

$$RN_g(x) = g'\big(F_X(x)\big) = \exp\big(\lambda \cdot \Phi^{-1}(F_X(x))\big) \cdot \exp\left(-\lambda^2/2\right).$$

Figure 3.1. Radon-Nikodym Derivative Implied By The Wang Transform



One interesting property is that both normal and lognormal distributions are preserved under the Wang transform in equation (3.2):

- If F has a Normal(μ,σ^2) distribution, F^* is also a normal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$.
- If F has a log-normal(μ,σ^2) distribution such that $\ln(X) \sim$ Normal(μ,σ^2), F^* is another log-normal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$.

4. Normalized Exponential Tilting and the Wang transform

Theorem 4.1: Assume that X and Y have bivariate normal copula with a correlation coefficient of $\rho_{X,Y}$. The normalized exponential tilting in equation (2.3) is equivalent to applying the Wang transform:

$$F_X^*(x) = g(F_X(x)) = \Phi\left[\Phi^{-1}(F_X(x)) - \beta\right], \quad \text{with } \beta = \rho_{X,Y} \cdot \lambda .$$

A proof can be found in Wang (2003).

Theorem 4.1 establishes an important link between normalized exponential tilting and the Wang transform. This result also recovers of the Capital Asset Pricing Modeling, which reveals the meaning of the λ parameter.

Consider a stock index R_M (represent the market portfolio) whose prospective end-of-period return has a normal (μ_M, σ_M^2) distribution with mean μ_M and standard deviation σ_M. The discounted end-of-period stock price

$$S_M(1)\cdot\exp(-r) = S_M(0)\cdot\exp(R_M)\cdot\exp(-r)$$

has a log-normal$\left(\mu_M - r - \frac{\sigma_M^2}{2}, \quad \sigma_M^2\right)$ distribution, and r is the risk-free rate of return.

If we apply normalized exponential tilting on $X= R_M$ with the reference Y being the stock return R_M, the risk-neutralized distribution for the stock return has a normal distribution with $E^*[R_M] = \mu_M - \lambda_M \cdot \sigma_M$. By forcing the risk-neutralized expected return equal the risk free rate, r, we get,

$$\lambda_M = \frac{E[R_M] - r}{\sigma_M} = \frac{\mu_M - r}{\sigma_M}.$$

Alternatively, if we apply normalized exponential tilting on the stock price X= $S_M(1)$ with the reference $Y=X$ being the stock price, the risk-neutralized distribution for the stock price is log-normal$\left(\mu_M - \lambda\sigma_M - r - \frac{\sigma_M^2}{2}, \quad \sigma_M^2\right)$. By forcing the risk-neutralized expected value of the discounted stock price $S_M(1)$ to equal the current stock price $S_M(0)$, we get the same result[1]:

$$\lambda_M = \frac{E[R_M] - r}{\sigma_M} = \frac{\mu_M - r}{\sigma_M}.$$

[1] However, for the exponential tilting in equation (2.1) without the normalization procedure, using the stock price as reference Y would not yield another risk-neutralized lognormal stock price distribution.

For the stock index, the risk adjustment parameter λ is exactly the market price of risk (or the Sharpe ratio). This special case helps us to assign a definite meaning to the parameter λ, as an extension of the market price of risk (or Sharpe ratio) to risks with general-shaped distributions.

Now we consider an asset *i* on the same one-period time horizon. Let $X=R_i$ be the return for asset *i*. Assume that that $\{R_i, R_M\}$ follow a normal copula with correlation coefficient $\rho_{i,M}$. Applying normalized exponential tilting on $X=R_i$ with the reference *Y* being *either* the stock return R_M *or* the stock price $S_M(1)$, Theorem 4.1 yields that

$$\lambda_i = \rho_{i,M} \cdot \lambda_M \text{ , or equivalently}$$

$$\frac{E[R_i]-r}{\sigma_i} = \rho_{i,M} \cdot \frac{E[R_M]-r}{\sigma_M}.$$

This is exactly the CAPM result for the expected return of stock *i* in relation to the market portfolio. Thus, Theorem 4.1 extends CAPM to cases that $\{R_i, R_M\}$ follow a normal copula without restriction on the shape of their marginal distributions.

5. Valuation of Contingent Claims

If $X=h(Y)$ be a (monotone) function of the variable *Y*, we say that *X* is a (monotone) contingent claim of the underlying risk *Y*. The market price of risk for a monotone contingent claim $X=h(Y)$ is the same as that for the underlying risk *Y*.

Theorem 5.1. When valuing a monotone contingent claim $X=h(Y)$, the following risk-neutralization methods are equivalent when the same λ value is used:

1) Apply normalized exponential tilting of *Y* w.r.t. *Y*, and calculate the expected value of $X=h(Y)$ under the risk-neutralized distribution of the underlying risk *Y*.
2) Apply normalized exponential tilting of *X* w.r.t. *Y*, and calculate the expected value of *X* under the risk-neutralized distribution of *X*.
3) Apply Wang transform to the CDF of *Y*, and calculate the expected value of $X=h(Y)$ under the risk-neutralized distribution for the underlying risk *Y*.

4) Applying Wang transform to the CDF of *X*, and calculate the expected value of *X* under the risk-neutralized distribution of *X*.

Consider the special case that $X = \max\{0, \quad Y - K\}\exp(-r)$, where *Y* represents the end-of-period stock price variable which has a lognormal distribution. *X* is the payoff of a European call option on *Y* with a strike price *K*. When valuing this contingent claim using the normalized exponential tilting with λ being the stock's market price of risk, we recover the Black-Merton-Scholes formula for European call options (more detailed derivations is given in Wang, 2000).

In the remaining of this paper we shall extend the normalized exponential tilting to multivariate cases.

6. Normalized Multivariate Exponential Tilting

Consider *n* variables $\{X_1, X_2, ..., X_n\}$ and *k* references $\{Y_1, Y_2, ..., Y_k\}$ on a probability space (Ω, P).

Definition 6.1. For each scenario ω in the probability space (Ω, P), the exponential tilting of $\{X_1, X_2, ..., X_n\}$ with respect to references $\{Y_1, Y_2, ..., Y_k\}$ is defined by the following p.d.f.:

$$(6.1) \qquad \frac{f^*(x_1(\omega), x_2(\omega), \cdots, x_n(\omega))}{f(x_1(\omega), x_2(\omega), \cdots, x_n(\omega))} = c \cdot E\left[\exp\left(\sum_{j=1}^{k} \lambda_j Y_j(\omega)\right)\right].$$

where $\{\lambda_1, \lambda_2, ..., \lambda_k\}$ are real-valued parameters that control the magnitude of risk-adjustment, and *c* is a normalizing coefficient.

In terms of joint probability density function, we can reformulate equation (6.1) as follows:

$$\frac{f^*(x_1, x_2, \cdots, x_n)}{f(x_1, x_2, \cdots, x_n)} = c \cdot E\left[\exp\left(\sum_{j=1}^{k} \lambda_j Y_j\right) \;\middle|\; X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\right].$$

In definition 6.1 we leave much flexibility in the choice of the references $\{Y_1, Y_2, ..., Y_k\}$. For instance, one can choose the references $\{Y_1, Y_2, ..., Y_k\}$ to be the risks $\{X_1, X_2, ..., X_n\}$ themselves, the company aggregate, or some industry indices.

In order to get a meaningful interpretation of the parameters $\{\lambda_1, \lambda_2, ..., \lambda_k\}$, we need to apply the normalization procedure to all references $\{Y_1, Y_2, ..., Y_k\}$.

Definition 6.2 Assume that there exist standard normal variables $\{Z_1, Z_2, ..., Z_k\}$ such that

$$Y_1 = F_{Y_1}^{-1}(\Phi(Z_1)), \quad Y_2 = F_{Y_2}^{-1}(\Phi(Z_2)), \quad \cdots, \quad Y_k = F_{Y_k}^{-1}(\Phi(Z_k))$$

We define the normalized multivariate exponential tilting of $\{X_1, X_2, ..., X_n\}$ with respect to references $\{Y_1, Y_2, ..., Y_k\}$ as the following:

(6.2) $$\frac{f^*(x_1, x_2, \cdots, x_n)}{f(x_1, x_2, \cdots, x_n)} = c \cdot E\left[\exp\left(\sum_{j=1}^{k} \lambda_j Z_j\right) \mid X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\right]$$

7. Multivariate Distortions

Consider multivariate risks $\{X_1, X_2, \cdots, X_n\}$ that have marginal CDFs

$\{F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n)\}$, respectively.

Assume that $\{X_1, X_2, \cdots, X_n\}$ have a joint CDF specified by

$$F_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = C(F_{X_1}(x_1), \quad F_{X_2}(x_2), \quad \cdots, \quad F_{X_n}(x_n)),$$

where $C(\ldots)$ is a multivariate uniform distribution or a copula function (see Embrechts et al, 2002).

Definition 7.1 We define *separate* distortions $\{g_1, g_2, \cdots, g_n\}$ such that the resulting multivariate distribution has the following marginal distributions:

$$F_{X_j}^*(x_j) = g_j[F_{X_j}(x_j)],$$

and the same correlation structure in term of copula function:

$$F_{X_1, X_2, \cdots, X_n}^*(x_1, x_2, \cdots, x_n) = C(F_{X_1}^*(x_1), \quad F_{X_2}^*(x_2), \quad \cdots, \quad F_{X_n}^*(x_n)).$$

Definition 7.2 We define *joint* distortions $\{g_1, g_2, \cdots, g_n\}$ in terms of the joint p.d.f.:

$$f^*_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = RN_{g_1, g_2, \cdots, g_n}(x_1, x_2, \cdots, x_n) \cdot f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n),$$

where the Radon-Nikodym derivative is given by:

- In the discrete case for the point $\vec{\mathrm{x}}_i = (x_{1;i}, x_{2;i}, \cdots, x_{n;i})$ we have

$$RN_{g_1, g_2, \cdots, g_n}(x_{1;i}, x_{2;i}, \cdots, x_{n;i}) = c \cdot \prod_{j=1}^{n} \frac{g_j\left(F_{X_j}(x_{j;i})\right) - g_j\left(F_{X_j}(x_{j;i-1})\right)}{F_{X_j}(x_{j;i}) - F_{X_j}(x_{j;i-1})}$$

- In the continuous case for the point $\vec{\mathrm{x}} = (x_1, x_2, \cdots, x_n)$ we have

$$RN_{g_1, g_2, \cdots, g_n}(x_1, x_2, \cdots, x_n) = c \cdot \prod_{j=1}^{n} g_j'\left(F_{X_j}(x_j)\right).$$

Theorem 7.1 When $\{X_1, X_2, \cdots, X_n\}$ have uncorrelated marginal distributions, both the separate distortions and the joint distortions yield the same risk-neutralized multivariate probability distribution with uncorrelated marginal distributions

$$f^*(x_1, x_2, \cdots, x_n) = \prod_{j=1}^{n} f^*_{X_j}(x_j), \text{ and}$$

$$F^*_{X_j}(x_j) = g_j[F_{X_j}(x_j)], \text{ with } j = 1,2,\ldots, n.$$

When $\{X_1, X_2\}$ are correlated, the separate distortions and the joint distortions can yield different results. Joint distortions reflect the interactions between X_1 and X_2 in the probability adjustment, while separate distortions do not. Consider the special case that $X_1 = X_2$, and $g_1 = g_2$ are the Wang transform with parameter λ. Under the joint distortions $\{g_1, g_2\}$, the risk-neutralized distribution for X_1 is equivalent to applying the Wang transform to X_1 with parameter 2λ. In contrast, under the separate distortions $\{g_1, g_2\}$, the risk-neutralized distribution for X_1 is equivalent to applying the Wang transform to X_1 with parameter λ.

When $g_j(u) = \Phi\left[\Phi^{-1}(u) + \lambda_j\right]$, for *j=1, 2, ..., n*, we shall refer to the separate distortions in definition 7.1 as *separate Wang transforms* with parameters $\{\lambda_1, \lambda_2, ..., \lambda_n\}$; and we shall refer to the joint distortions in definition 7.2 as *joint Wang transforms* with parameters $\{\lambda_1, \lambda_2, ..., \lambda_n\}$.

8. Link between Exponential Tilting and Distortion – the Multivariate Case

Theorem 8.1 Assume that the *n* variables and the *k* references

$$\{X_1, X_2, \cdots, X_n; Y_1, Y_2, \cdots, Y_k\}$$

follow a normal copula with correlation matrix:

$$\Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} & \cdots & \rho_{X_1,X_n} \\ \rho_{X_1,X_2} & 1 & \cdots & \rho_{X_2,X_n} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{X_1,X_n} & \rho_{X_2,X_n} & \cdots & 1 \end{pmatrix}.$$

The normalized multivariate exponential tilting (6.2) of $\{X_1, X_2, \cdots, X_n\}$ w.r.t. $\{Y_1, Y_2, \cdots, Y_k\}$ is equivalent to applying *separate* Wang transforms to X_i with:

$$g_i(u) = \Phi\left[\Phi^{-1}(u) + \beta_i\right], \text{ and } \beta_i = \sum_{j=1}^{k} \rho_{X_i,Y_j} \cdot \lambda_j, \text{ (for } i=1, 2, ..., n).$$

The correlation matrix between $\{X_1, X_2, \cdots, X_n\}$ is unchanged after the normalized multivariate exponential tilting, $\Sigma^* = \Sigma$.

Remark: In his independent work, Professor M. Kijima (2005) has derived this multivariate Wang transform using an equilibrium model.

Example 8.1. Assume that the risks $\{X_1, X_2\}$ have a bivariate normal(0,1) with correlation coefficients:

$$\Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

According to Theorem 8.1, under bivariate normalized exponential tilting (6.2) with references $Y_1 = X_1$ and $Y_2 = X_2$, the risk-neutralized joint distribution for $\{X_1^*, X_2^*\}$ is also bivariate normal with correlation coefficients:

$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

For illustration, we choose $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$. The risk-neutralized marginal distributions are equivalent to applying *separate* Wang transforms $F_{X_j}^*(x) = \Phi\left[\Phi^{-1}(F_{X_j}(x)) - \beta_j\right]$, for j =1, 2, with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \rho_{X_1,X_2}\lambda_2 \\ \rho_{X_1,X_2}\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.38 \end{pmatrix}.$$

Figure 8.1 Scatter plot bivariate variables $\{X_1, X_2\}$, and Radon-Nikodym derivatives $RN(x_1, x_2)$

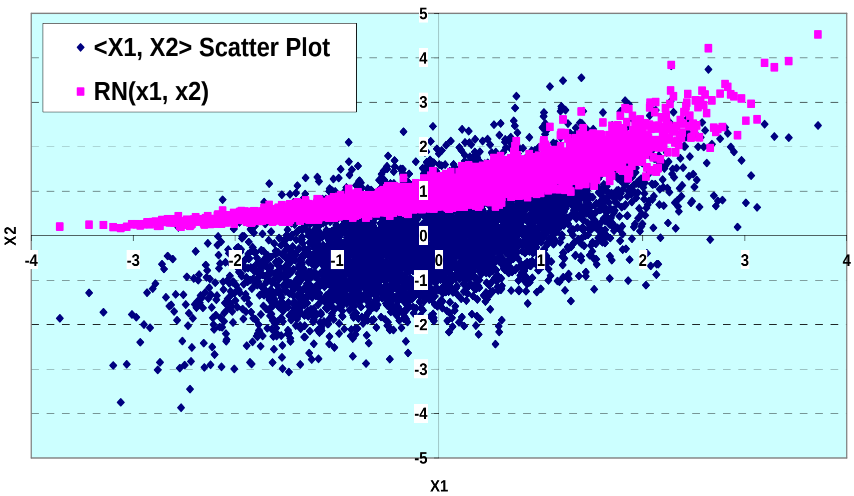


Figure 8.1 shows a scatter plot of $\{X_1, X_2\}$ and their corresponding Radon-Nikodym derivatives. The right-most blue diamond is a scatter plot of (x_1=3.195, x_2=2.505). The right-most red square gives its value of Radon-Nikodym derivative RN(3.195, 2.505)=3.884. One can see that the Radon-Nikodym

derivatives increase exponentially when the point (x_1, x_2) moves from the lower left quadrant to the upper right quadrant.

9. Valuing Contingent Claims on Multiple Underlying Risks

Consider contingent claims on multiple underlying variables:

$$X_i = h_i(Y_1, Y_2, \cdots, Y_k),\ i=1, 2, \ldots, n.$$

When valuing the contingent claim $X_i = h_i(Y_1, Y_2, \cdots, Y_k)$, the market price of risk should be specified through the underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

Theoretically, we should first adjust the multivariate probability measure for the underlying risks, and then value contingent claims as expected payoff under the risk-neutralized probability measure. Accordingly, we should first apply normalized exponential tilting of $\{Y_1, Y_2, \cdots, Y_k\}$ w.r.t. themselves, and calculate the expected value of $X_i = h_i(Y_1, Y_2, \cdots, Y_k)$ under the risk-neutralized distribution of the underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

Theorem 9.1. If we let $X_j=Y_j$ be the underlying risks themselves, for j=1, 2, ..., k, The normalized multivariate exponential tilting (6.2) of $\{Y_1, Y_2, \cdots, Y_k\}$ w.r.t. themselves is equivalent to applying joint Wang transforms with parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

This result has implications in pricing contingent claims on multiple underlying risks $\{Y_1, Y_2, \cdots, Y_k\}$.

Example 9.1. Applications in Pricing Contingent Claims. Suppose that the underlying risks, (Y_1, Y_2), have the following empirical bivariate distribution. Although the linear correlation coefficient between (Y_1, Y_2) can be estimated as 0.38, their correlation structure is nowhere near a normal copula.

Scenario	Y_1	Y_2
1	-	1,954.08
2	-	2,239.22
3	-	2,974.21
4	-	3,275.38
5	-	3,351.93
6	-	6,526.96
7	-	9,542.63
8	-	13,999.95
9	-	14,279.63
10	-	14,519.32
11	-	16,179.92
12	-	19,134.14
13	-	35,071.98
14	-	57,591.43
15	-	62,967.38
16	-	82,638.17
17	-	248,909.05
18	638.80	3,331.31
19	1,533.11	2,047.14
20	5,110.36	1,159.07
21	6,387.95	2,152.74
22	6,387.95	8,940.58
23	8,943.13	4,949.35
24	11,498.32	-
25	15,331.09	-
26	27,279.12	-
27	35,772.54	5,634.79
28	93,264.12	24,115.73
29	102,207.25	6,287.06
30	191,638.60	34,096.74
31	246,010.43	232,641.59
32	511,036.26	39,161.24
33	511,036.26	150,301.30
34	662,650.50	73,140.35

- Contract #1 has a contingent payoff in the amount of Y_1 in excess of 200,000. That is, the payoff $X_1 = \max\{Y_1 - 200000, \quad 0\}$.
- Contract #2 has a contingent payoff of 50% of the amount of Y_2. That is, $X_2 = 0.5Y_2$
- Contract #3 has a contingent payoff in the amount of Y_1 in excess of 200,000, plus 50% of the amount of Y_2. Technically, Contract 3 is simply the combination of Contract #1 and Contract #2: $X_3 = X_1 + X_2$.

Without risk-adjustment, the expected payoffs for Contract #1, #2, #3 are \$33,257, \$17,399, \$50,656, respectively.

Suppose that the market price of risk for the underlying risks Y_1 and Y_2 are λ_1=0.3 and λ_2=0.2, respectively. We derive a risk-neutralized bivariate distribution by applying normalized bivariate exponential tilting of Y_1 and Y_2 with respect to themselves, using λ_1=0.3 and λ_2=0.2.

Theorem 9.1 facilitates a numerical method for calculating the risk-neutralized probabilities for each of the 34 scenarios. Based on the risk-neutralized probabilities for each scenario, we calculated the prices for Contract #1, #2, #3 being \$68,240, \$24,847, and \$93,087 respectively.

	Expected Payoff of Contract #1	Expected Payoff of Contract #2	Expected Payoff of Contract #3
No Risk Adjustment	\$ 33,257	\$ 17,399	\$ 50,656
With Risk Adjustment	\$ 68,240	\$ 24,847	\$ 93,087
Loading	105%	43%	84%

Note that the obtained prices are additive. Indeed, the only way to ensure price additivity is through a change of bivariate probability measure.

Example 9.2 Numerical Techniques Involving Discrete Distributions.

Consider the following bivariate distribution (that does not follow a normal copula).

	X_2=1	X_2=2	X_2=3	X_2=4	X_2=5
X_1=1	*0.20*	*0.07*	*0.06*	*0.05*	*0.04*
X_1=2	*0.06*	*0.05*	*0.04*	*0.03*	*0.03*
X_1=3	*0.05*	*0.04*	*0.03*	*0.03*	*0.02*
X_1=4	*0.03*	*0.03*	*0.02*	*0.02*	*0.01*
X_1=5	*0.03*	*0.02*	*0.01*	*0.02*	*0.01*

We want to compute the adjusted joint distribution for the normalized multivariate exponential tilting of (X_1, X_2), with reference to themselves, and with λ_1=0.3 and λ_2=0.2.

We first apply the Wang transform to X_1 with λ_1=0.3.

$X_1 = x_1$	$f(x_1)$	$F(x_1)$	$F^*(x_1)$	$f^*(x_1)$
1	0.42	0.42	0.30787	0.30787
2	0.21	0.63	0.51271	0.20483
3	0.17	0.80	0.70596	0.19325
4	0.11	0.91	0.85101	0.14505
5	0.09	1.00	1.00000	0.14899

We then apply the Wang transform to X_2 with λ_2=0.2.

$X_2 = x_2$	$f(x_2)$	$F(x_2)$	$F^*(x_2)$	$f^*(x_2)$
1	0.37	0.37	0.29741	0.29741
2	0.21	0.58	0.50076	0.20334
3	0.16	0.74	0.67124	0.17049
4	0.15	0.89	0.84768	0.17644
5	0.11	1.00	1.00000	0.15232

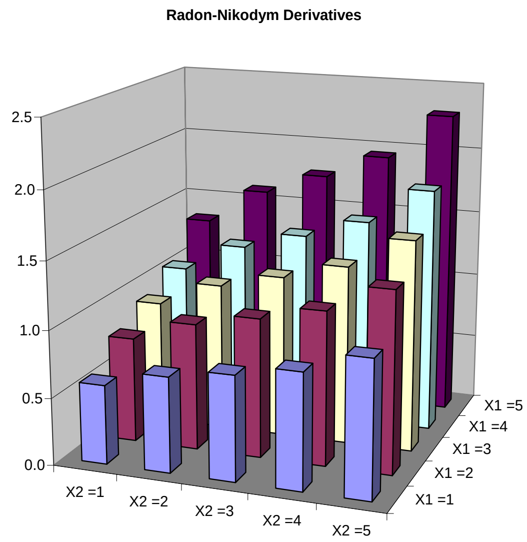
According to Theorem 9.1, the bivariate Radon-Nikodym derivatives are:

$$RN_g(x_1, x_2) = \frac{f^*_{X_1,X_2}(x_1, x_2)}{f_{X_1,X_2}(x_1, x_2)} = c \cdot \frac{f^*_{X_1}(x_1)}{f_{X_2}(x_1)} \cdot \frac{f^*_{X_2}(x_2)}{f_{X_2}(x_2)}.$$

The final risk-neutralized joint probability (density) function is:

	X_2=1	X_2=2	X_2=3	X_2=4	X_2=5
X_1=1	0.1178	0.0497	0.0469	0.0431	0.0406
X_1=2	0.0470	0.0472	0.0416	0.0344	0.0405
X_1=3	0.0457	0.0440	0.0363	0.0401	0.0315
X_1=4	0.0318	0.0383	0.0281	0.0310	0.0183
X_1=5	0.0399	0.0321	0.0176	0.0389	0.0229

Figure 9.1. The Bivariate Radon-Nikodym Derivatives



As shown in Figure 9.1, the Radon-Nikodym derivative increases to its highest value at $\{X_1=5, X_2=5\}$, indicating the largest relative risk adjustment at the joint tail of the bivariate variables.

10. Conclusion & Future Research

We have introduced the concept of normalized exponential tilting, and established an important link with probability distortions, first for the univariate case, then for the multivariate case. Normalized exponential titling provides a general framework for pricing risks with respect to given reference risks, and for valuing contingent claims on given underlying risks. The paper also provides efficient numerical routines for risk-neutralizing multivariate probability distributions.

One area of future research is to investigate other plausible normalization procedures on the reference risk from an economic point of view, and perform empirical tests based on market data. Note that some previous empirical studies have been reported in Madan and Unal (2004) and Wang (2004). Another area of future research is to investigate to what extent the interactions among multiple reference risks can impacts the measure-change density in the normalized multivariate exponential tilting.

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